Computation of PARMA Densities and a PARMA Representation of a PARMA Sequence

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Content

Abstract

In my talk given on-line at The XIII Workshop on Nonstationary Systems, Grodek nad Dunajcem 2020, I showed how the results from the paper

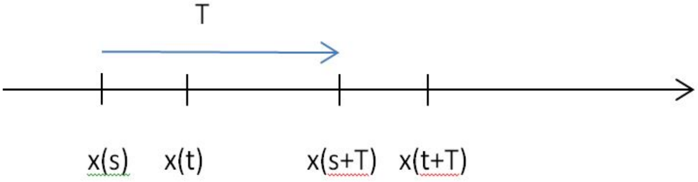
Makagon, A. "Periodically Corerlated Sequences with Rational Spectra and PARMA Systems." in: Contributions to the 9th Workshop on Cyclostationary Systems and Their Applications, Grodek, Poland, 2016

lead to computationally explicit procedures to compute the density and a representation of a PARMA sequence. Here I want to elaborate on computational difficulties involved, as well to show some applications of the procedures.

PC Sequences

A sequence of complex finite variance and zero mean random variables $x(n)$, $n \in Z$, is called *periodically correlated with period T (T-PC)* if

$$R_x(s+T, t+T) = Ex(s+T)\overline{x(t+T)} = Ex(s)\overline{x(t)} = R_x(s,t).$$



Define

$$a_j(n) := \sum_{r=0}^{T-1} e^{-ijr2\pi/T} R_x(n+r, r), \qquad j = 0, \ldots, T-1.$$

Transfer Function of a PC Sequence

Spectrum of $(x(n))$ is a complex C^T-values vector measure $\gamma(dt) = (\gamma^0(dt), \ldots, \gamma^{T-1}(dt))$ defined on $[0, 2\pi)$ such, that

$$a_j(n) = \int_0^{2\pi} e^{-itn} \gamma^j(dt), \quad n \in Z.$$

If there is a function $g(z) = (g^0(z), \ldots, g^{T-1}(z))$ of a complex variable $z \in D = \{z \in C : |z| = 1\}$ such that

$$\gamma(dt) = (1/2\pi) g(e^{it}) dt, \quad t \in [0, 2\pi),$$

then $(x(n))$ is called a.c. and $g(e^{it})$ is called a density of $(x(n))$
Any C^T valued function $h(z) = (h^0(z), \ldots, h^{T-1}(z))$, $z \in D$, such that

$$g^j(e^{it}) = h(e^{it}) h(e^{i(t+2\pi j/T)})^*$$

is called a *transfer function* of $(x(n))$ [Makagon, Miamee 2013]

Corresponding Stationary Sequence

There is one-to one correspondence between T-PC sequences and T-variate stationary sequences

$$T - PC \ni (x(n)) \longleftrightarrow (\mathbf{x}(n)) \in T\text{-variate stationary}$$

$$\ldots, x(-1), \underbrace{x(0), x(1), \ldots, x(T-1)}_{\mathbf{x}(0)^t}, \underbrace{x(T), \ldots, x(2T-1)}_{\mathbf{x}(1)^t}, x(2T), \ldots$$

$$\mathbf{x}(n) == \begin{bmatrix} x(nT) \\ x(nT+1) \\ \vdots \\ x((n+1)T-1) \end{bmatrix}$$

Transfer Function of a Stationary Sequence

If $\mathbf{x}(n) = [x^k(n)], n \in Z$ is T -variate stationary, then a $T \times T$-matrix function $F(z) = [F^{j,k}(z)]$, $z \in D$ such that

$$Ex^j(n)\overline{x^k(0)} = \int_0^{2\pi} e^{-int} F^{j,k}(e^{it})dt, \quad n \in \mathcal{Z}, 0 \leq j, k \leq T-1,$$

called a spectral density of $(\mathbf{x}(n))$. Given $F(e^{it})$, any $T \times T$-matrix function $H(z)$, $z \in D$, such that

$$F(e^{it}) = H(e^{it})H(e^{it})^*$$

is called a *transfer function* of $(\mathbf{x}(n))$.

Relations between transfer functions of $(x(n))$ and the corresponding $(\mathbf{x}(n))$ were found in [Makagon 2017]

PARMA System

A system of difference equations

$$x(n) = -\sum_{j=1}^{l} a_j(n)x(n-j) + \sum_{j=0}^{r} b_j(n)\xi_{n-j}, \qquad n \in Z,$$

is called a *PARMA system of period T* if $a_j(n), b_j(n)$ are T-periodic and (ξ_n) are zero mean uncorrelated with unit variance. It is convenient to write it as

$$\sum_{j=0}^{l} a_j(n)x(n-j) = \sum_{j=0}^{r} b_j(n)\xi_{n-j}, \qquad a_0(n) = 1, \tag{1}$$

A *PARMA sequence* $(x(n))$ is an a.c. T-PC solution to the equation (1) (if exists). A system (1) is called a *representation of a PARMA sequence* $(x(n))$ if for every n

$$M_x(n) = sp\{x(m) : m \leq n\} = M_\xi(n) = sp\{\xi_m : m \leq n\}$$

Corresponding VARMA System

If we arrange the coefficients $a_j(n)$ in (1) into an $T \times (L+1)T$ matrix $[A_L \ \dots \ A_1 \ A_0]$ where L is such that the matrix contains all nonzero $a_j(n)$'s as shown below

$$\left[\begin{array}{c|ccc|cccc} \dots & a_T(0) & \dots & a_1(0) & a_0(0) & 0 & \dots & 0 \\ \dots & a_{T+1}(1) & \dots & a_2(1) & a_1(1) & a_0(1) & \dots & 0 \\ \dots & \dots & \mathbf{A(1)} & \dots & \dots & \mathbf{A(0)} & \dots & \dots \\ \dots & \dots & \dots & a_T(T-1) & a_{T-1}(T-1) & \dots & \dots & a_0(T-1) \end{array}\right],$$

(here $a_0(j) = 1$) do the same for $b_j(n)$'s obtaining $[B_R \ \dots \ B_1 \ B_0]$, and denote $(\mathbf{x}(n))$ and (ξ_n) to be the corresponding T-variate stationary sequences then a PARMA system (1) can be written as a VARMA system

$$\sum_{j=0}^{L} A_j \mathbf{x}(n-j) = \sum_{j=0}^{R} B_j \xi_{n-j}, \qquad n \in \mathcal{Z},$$

Associated Matrix Polynomials

A PARMA system is therefore a VARMA system for which A_0 is lower triangular and has and unit diagonal and B_0 is lower triangular.
A VARMA (and hence PARMA) system is completely described by two polynomial matrices

$$A(z) = \sum_{k=0}^{L} A_k z^k, \quad B(z) = \sum_{k=0}^{R} B_k z^k.$$

VARMA systems were studied in many publications and books on Time Series Analysis. Existence and properties of a solution to a VARMA system depend on zeros of polynomial matrices $A(z)$ and $B(z)$ (for example [Hannan and Deistler 2012]). A zero of matrix polynomial is a number z for which the matrix drops its rank. In this talk we will be assuming that $\det(A(z_1)) \neq 0$ and $\det(B(z_2)) \neq 0$ for some $z_1, z_2 \in D$.

Computing PARMA Densities

Procedure: Given PARMA system (1)

- compute $A(z) = \sum_{k=0}^{L} A_k z^k$ and $B(z) = \sum_{k=0}^{R} B_k z^k$;
- if $\det(A(z) = 0$ for some z of modulus one, then the system has no unique a.c. T-PC solution;
- otherwise we compute $H(z) = A(z)^{-1} B(z)$ (this is a transfer function of $(\mathbf{x}(n))$);
- compute $h(z) = (1, z, \ldots z^{T-1}) H(z^T)$ (this is a transfer function of $(x(n))$, [Makagon 2017, Lemma 3.1 (1)]);
- compute $g^j(e^{it}) = h(e^{it}) h(e^{i(t+2\pi j/T)})^*$, $j = 0, \ldots, T-1$.

The function $g(e^{it}) = (g^0(e^{it}), ..., g^{T-1}(e^{it}))$ is the density of an a.c T-PC solution to the PARMA system (1)

Example 1

Consider a system

$$\begin{aligned} X(0) &= X(-1) + \xi_0 \\ X(1) &= 4X(0) + \xi_1 - \xi_{-1} \\ X(2) &= 2X(1) + \xi_2 + 2\xi_1. \end{aligned}$$

Associated matrix polynomials are

$$A(z) = \begin{bmatrix} 1 & 0 & -z \\ -4 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}, \quad B(z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -z \\ 0 & 2 & 1 \end{bmatrix},$$

$$H(z) = A(z)^{-1}B(z) = \frac{1}{8z-1} \begin{bmatrix} -1 & -4z & 2z^2 - z \\ -4 & -8z - 1 & -3z \\ -8 & -4 & 2z - 1 \end{bmatrix},$$

Example 1: cont.

A transfer function $h(z) = (1, z, \ldots z^{T-1})H(z^T)$ of $(x(n))$ is therefore
$h(z) = \frac{1}{8z^3-1}\left(-8z^2 - 4z - 1, -8z^4 - 4z^3 - 4z^2 - z, 2z^6 + 2z^5 - 3z^4 - z^3 - z^2\right),$
and the density $g(e^{it}) = (g^0(e^{it}), ..., g^{T-1}(e^{it}))$ of an a.c. T-PC solution to this system is

$$
\begin{aligned}
g^0(e^{it}) &= \frac{-(156\cos(t)+188\cos(t)^2+32\cos(t)^3-32\cos(t)^4+115)}{(16\cos(3t)-65)} \\
g^1(e^{it}) &= \frac{(66\cos(2t)+20\cos(3t)+2\cos(4t)+66\sqrt{3}\sin(2t)-2\sqrt{3})\sin(4t)+131)}{(32\cos(3t)-130)} \\
&+ i\frac{-(24\sin(2t)+6\sin(4t)+54\sin(t)-5\sqrt{3}-18\sqrt{3}\cos(t)+8\sqrt{3}\cos(2t)-20\sqrt{3}\cos(3t)-2\sqrt{3}\cos(4t))}{(32\cos(3t)-130)} \\
g^2(e^{it}) &= \frac{(66\cos(2t)+20\cos(3t)+2\cos(4t)-66\sqrt{3}\sin(2t)+2\sqrt{3}\sin(4t)+131)}{(32\cos(3t)-130)} \\
&- i\frac{(24\sin(2t)+6\sin(4t)+54\sin(t)+5\sqrt{3}+18\sqrt{3}\cos(t)-8\sqrt{3}\cos(2t)+20\sqrt{3}\cos(3t)+2\sqrt{3}\cos(4t)}{(32\cos(3t)-130)}
\end{aligned}
$$

Example 1: graphs

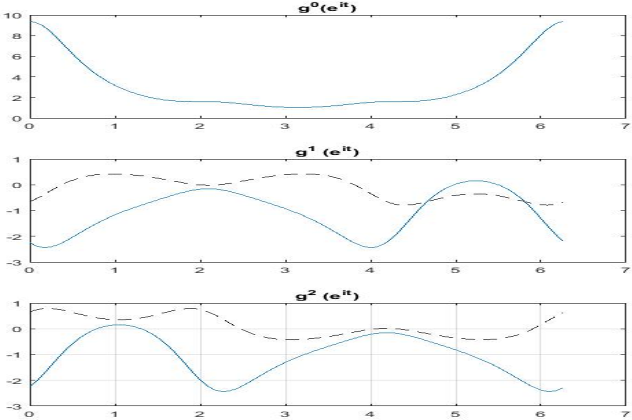


Figure: Continuous line = $Re(g^j)$, dashed line = $Im(g^j)$

Constructing PARMA Representation

Suppose that a full-rank T-PC sequence $(x(n))$ has a rational density g. Only sequences with rational density admit a PARMA representation [Makagon 2017, Theorem 3.1]. In order to find a PARMA representation of $(x(n))$ we do the following:

- STEP 1: express $g(e^{it}) = (g^0(e^{it}), \dots, g^{T-1}(e^{it}))$ in terms of $z = e^{it}$ and compute the density $F(z)$ of the corresponding T-variate stationary sequence $(\mathbf{x}(n))$ as follows [Makagon 2017, Corollary 3.2]

$$F^{j,k}(z^T) = (1/T^2) z^{k-j} \sum_{p=0}^{T-1} \sum_{q=0}^{T-1} d^{kq-jp} g^{\langle q-p \rangle}(zd^p).$$

  where $z = e^{it}$, $d = e^{i2\pi/T}$ and $\langle m \rangle$ is the reminder in division of m by T.

Constructing PARMA Representation: cont.

- STEP 2: find two left coprime polynomial matrices $A_1(z)$ and $B_1(z)$ with *no zeros in the open unit disk*, such that if we denote $\Gamma(z) = A_1(z)^{-1}B_1(z)$, then

$$F(e^{it}) = \Gamma(e^{it})\,\Gamma(e^{it})^*, \quad t \in [0, 2\pi)$$

  The system $(A_1(z), B_1(z))$ is a VARMA representation of the corresponding stationary sequence $(\mathbf{x}(n))$,
- STEP 3: find a constant unitary matrix V such that $B(0) = A_1(0)^{-1}B_1(0)V$ is lower triangular with nonnegative diagonal entries

Then the system $(A(z), B(z))$, where $A(z) = A_1(0)^{-1}A_1(z)$ and $B(z) = A_1(0)^{-1}B_1(z)V$, is a PARMA representation of the T-PC sequence with density $g(e^{it})$.

Matrix Riesz-Fisher Theorem

The fundamental computational problem is STEP 3. Existence such polynomial matrices follows from matrix version Riesz-Fisher theorem proved by Rozanov.

[Rozanov 1967, Thm. 10.1] *Each a.e. nonegative rational square matrix function* $F(e^{it})$ *on* $[0, 2\pi)$ *of rank* r *can be represented in the form* $F(e^{it}) = \Gamma(e^{it})\Gamma(e^{it})^*$ *a.e. where* $\Gamma(z)$ *is rational and the rank of* $\Gamma(z)$ *is* r *for* all z *inside the open unit circle* $D_{<1} = \{|z| < 1\}$.

Construction of such $G(z)$ was given in [Rozanov 1967, Thm. 10.1] and later in [Hannan and Deistler 2012, p. 27]. Exact computation of $G(z)$ is hopeless. In applied Linear Systems and MIMO Analysis there are some papers attempted to provide stable numerical algorithms to find $G(z)$. Recently Geronimo, Woerdeman and Chung (2021) gave a different proof of Matrix Riesz-Fisher Theorem which gives hope for more stable algorithms.

Example 2

We will attempt to find a representation of the 3-PC sequence $(x(n))$ with density given in Example 1

$$
\begin{aligned}
g^0(e^{it}) &= \frac{-(156\cos(t)+188\cos(t)^2+32\cos(t)^3-32\cos(t)^4+115)}{(16\cos(3t)-65)}\\
g^1(e^{it}) &= \frac{(66\cos(2t)+20\cos(3t)+2\cos(4t)+66\sqrt{3}\sin(2t)-2\sqrt{3})\sin(4t)+131)}{(32\cos(3t)-130)}\\
&+ i\frac{-(24\sin(2t)+6\sin(4t)+54\sin(t)-5\sqrt{3}-18\sqrt{3}\cos(t)+8\sqrt{3}\cos(2t)-20\sqrt{3}\cos(3t)-2\sqrt{3}\cos(4t))}{(32\cos(3t)-130)}\\
g^2(e^{it}) &= \frac{(66\cos(2t)+20\cos(3t)+2\cos(4t)-66\sqrt{3}\sin(2t)+2\sqrt{3}\sin(4t)+131)}{(32\cos(3t)-130)}\\
&- i\frac{(24\sin(2t)+6\sin(4t)+54\sin(t)+5\sqrt{3}+18\sqrt{3}\cos(t)-8\sqrt{3}\cos(2t)+20\sqrt{3}\cos(3t)+2\sqrt{3}\cos(4t)}{(32\cos(3t)-130)}
\end{aligned}
$$

The system that yielded this density

$$
\begin{aligned}
X(0) &= X(-1)+\xi_0\\
X(1) &= 4X(0)+\xi_1-\xi_{-1}\\
X(2) &= 2X(1)+\xi_2+2\xi_1.
\end{aligned}
$$

is not a representation of $(x(n))$ since the zeros of system polynomials are 1/8 and -1/2, respectively (both inside the open unit disk)

Example 2 (cont)

STEP 1: First we compute the density $F(z)$ of the corresponding 3-variate stationary sequence

$$F(z) = \begin{bmatrix} \frac{(2z^2-22z+2)}{8z^2-65z+8} & \frac{2z^2-39z}{8z^2-65z+8} & \frac{2z^3-21z^2-6z}{8z^2-65z+8} \\ \frac{2-39z}{8z^2-65z+8} & \frac{-8z^2-90z-8}{8z^2-65z+8} & \frac{-35z^2-30z}{8z^2-65z+8} \\ \frac{-6z^2-21z+2}{8z^3-65z^2+8z} & \frac{-30z-35}{8z^2-65z+8} & \frac{2z^2-85z+2}{8z^2-65z+8} \end{bmatrix}$$

Then we attempted to find the matrix polynomials $A_1(z)$ and $B_1(z)$ as in STEP 2. We obtained only approximate solutions

$$A_1(z) = \begin{bmatrix} 2.0 & -2.18 & 0.76z + 2.42 \\ 0 & -4.0z - 41.9 & 29.8z + 94.9 \\ 0 & -424.0 & -8.0(1.0z - 29.4)(z + 3.19) \end{bmatrix}.$$

$$B_1(z) = \begin{bmatrix} 0.672z + 2.41 & 0.173z + 0.407 & 0.0407z - 0.444 \\ 22.1z + 99.7 & -2.36z - 5.78 & 1.29z - 14.0 \\ -1.95z^2 + 105.0z + 780.0 & -0.501z^2 - 9.03z + 1.6 & -0.118z^2 + 19.7z - 201.0 \end{bmatrix}.$$

Example 2 (cont)

STEP 3. Using Cholesky decomposition we find V and then an (approximate) PARMA representation $(A(z), B(z))$ of $(x(n))$

$$A(z) = \begin{bmatrix} 1 & -0.136z & 0.0475z^2 + 0.151z \\ 0 & 1.0 - 0.337z & 0.0853z^2 + 0.272z \\ 0 & -0.191z & 1.0 + 0.0376z^2 + 0.434z \end{bmatrix}$$

$$B(z) = \begin{bmatrix} 0.49 - 0.0803 * z & 0.0581 * z & 0.012 * z^2 + 0.464 * z \\ 0.953 - 0.144 * z & 0.283 * z + 0.514 & 0.0215 * z^2 + 0.687 * z \\ 0.211 - 0.0637 * z & 0.207 * z + 0.546 & 0.00948 * z^2 + 0.523 * z + 0.992 \end{bmatrix}$$

We can multiply any row of the concatenated matrix $[A(z), B(z)]$ by any polynomial with a zero constant term and add it to another row, or we can multiply any row of $[A(z), B(z)]$ by any polynomial and add it any of the following rows . In both cases $\Gamma(z) = A(z)^{-1}B(z)$ will be the same, and hence we obtain an equivalent representation (with possibly lower degrees of terms).

Example 2 (cont)

By doing this we obtained a "simpler" PARMA representation

$$A(z) = \begin{bmatrix} 1.0 & -0.14 * z & 0.047 * z^2 + 0.15 * z \\ -1.8 & 1.0 - 0.092 * z & 0 \\ -4.1 & 1.8 - 0.25 * z & 0.31 * z + 1.0 \end{bmatrix}$$

$$B(z) = \begin{bmatrix} 0.49 - 0.08 * z & 0.058 * z & 0.012 * z^2 + 0.46 * z \\ 0.073 & 0.18 * z + 0.51 & -0.15 * z \\ -0.044 & 0.49 * z + 1.5 & 0.99 - 0.11 * z \end{bmatrix}$$

The zeros of $A(z)$ are 10.9083633, -3.18614101, 8.000, and the zeros of $B(z)$ are 10.9083331, - 3.18614117, - 2.000, all outside the closed unit disk. The first two zeros of $(A(z)$ and $B(z))$ are very close and if we were able to perform exact computations they would have been removed in the process of dividing $(A(z)$ and $B(z))$ by the left greatest divisor.

Example 2 (cont)

In terms of difference equation a representation that we have obtained is

$x(0) = -0.0475x(-4) + 0.136x(-2) - 0.151x(-1) + 0.012\xi_{-4} - 0.0803\xi_{-3} + 0.0581\xi_{-2} + 0.464\xi_{-1} + 0.49\xi_0$

$x(1) = 0.0917x(-2) + 1.8x(0) + .178\xi_{-2} - 0.147\xi_{-1} + 0.0726\xi_0 + 0.514\xi_1$

$x(2) = 0.251x(-2) - 0.314x(-1) + 4.09x(0) - 1.84x(1) + 0.488\xi_{-2} - 0.114\xi_{-1} - 0.044\xi_0 + 1.49\xi_1 + 0.992\xi_2$

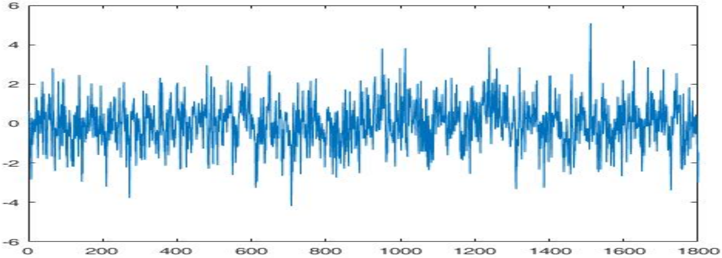
This representation is approximate and not the shortest. Approximation is very good, the *sup* distance between g and solution of the above system is less than 10^{-8}. Even not being the best, it is good enough for most of purposes, including prediction and simulation.

Simulation

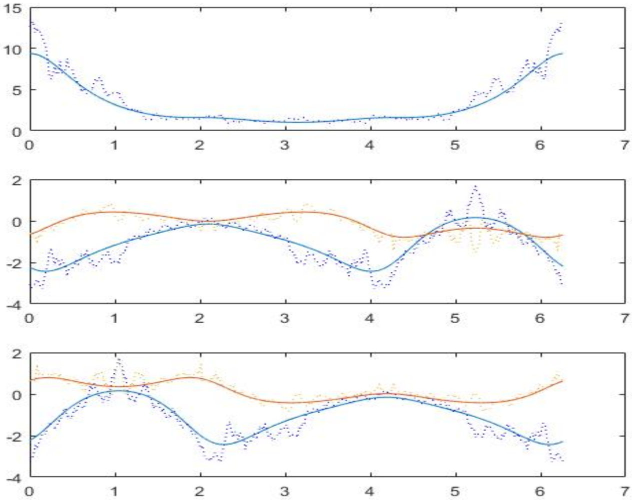
Knowing representation of a T-PC sequence $(x(n))$

$$x(n) = -\sum_{j=1}^{l} a_j(n)x(n-j) + \sum_{j=0}^{r} b_j(n)\xi_{n-j}, \qquad n \in Z,$$

allows us to simulate a trajectory of $(x(n))$, given its density. To see how it works we used representation from Example 2 to simulated a trajectory x_n, $n = 0, \ldots, 1799$ of a 3-PC sequence $(x(n))$ with density g given in Example 2.



Then we computed smoothed shifted periodogram [Hurd and Miamee, 2007] and graphed it together with g



Prediction

If $x(n) = -\sum_{j=1}^{l} a_j(n)x(n-j) + \sum_{j=0}^{r} b_j(n)\xi_{n-j}$, is a PARMA representation of a $(x(n))$, then then the best linear estimate of $x(n)$ based on the past

$$M_x(n-1) = sp\{x(m) : m \leq n-1\} = M_\xi(n-1)$$

i.e. the orthogonal projection of $x(n)$ onto $M_x(n-1)$, is

$$\hat{x}(n) = -\sum_{j=1}^{l} a_j(n)x(n-j) + \sum_{j=1}^{r} b_j(n)\xi_{n-j}, \qquad n \in Z.$$

Innovation Coefficients

Every T-PC sequence with rational density can be written in a unique way in terms of its *innovations*

$$x(n) = \sum_{j=0}^{\infty} c_j(n)\xi_{n-j}, \quad c_0(n) \geq 0.$$

Innovation coefficients $c_j(n)$ are T-periodic in n and $\sum_{j=0}^{k-1} |c_j(n)|^2$ is the variance of lag k prediction error of $x(n)$. Having a PARMA representation $(A(z), B(z))$ of $(x(n))$ we can compute the coefficients $c_k(n)$ as follows [Makagon 2017]: first compute $H(z) = A(z)^{-1}B(z)$ and $h(z) = (1, z, \dots z^{T-1})H(z^T)$ and then

$$(1/\sqrt{2\pi}) \int_0^{2\pi} e^{-it(j+k)} h^k(e^{it})dt = c_j(\langle j+k \rangle), \;\; j \geq 0, k = 0, \dots, T-1$$

where $\langle m \rangle$ is the reminder in division of m by T.

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