Self-equilibrated residual based error estimation on 1-irregular meshes in incompressible finite elasticity

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Outline

- Motivation: compare the top-valued error estimation techniques applied to nonlinear elasticity.
- Finite Element Method for nearly incompressible materials.
- The idea to estimate the error due to Rüter and Stein.
- Residual implicit error estimate due to Bank/Weiser, Demkowicz/Oden/Strouboulis.
- Residual implicit error estimate with self-equlibration of residuals. Equidistribution of fluxes:
  - Ladeveze and Maunder.
  - Ainsworth and Oden.
- Numerical examples.
- Generalization for constrained meshes.
- Numerical examples for irregular meshes.
- Summary and conclusions.

Formulation of finite elasticity bvp

Finite elasticity:

- $\boldsymbol{x}(\boldsymbol{X}, t)$ - Lagrangian description
- $\boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}}$ - deformation gradient
- $\boldsymbol{C} = \boldsymbol{F}^T \boldsymbol{F}$ - right Cauchy–Green tensor
- $J = \det(\boldsymbol{C})^{1/2} = \det(\boldsymbol{F}) > 0$ - volume ratio
- Multiplicative decomposition of deformation gradient (Flory 1961):

$$\bar{\boldsymbol{F}} = J^{-1/3}\boldsymbol{F}, \quad \bar{\boldsymbol{C}} = J^{-2/3}\boldsymbol{C}$$

- Invariants:

$$I_1 = tr(\boldsymbol{C}), \quad I_2 = \frac{1}{2}[(tr(\boldsymbol{C})^2 - tr(\boldsymbol{C}^2)], \quad I_3 = \det \boldsymbol{C}.$$

- Modified invariants:

$$\bar{I}_1 = tr(\bar{\boldsymbol{C}}), \quad \bar{I}_2 = \frac{1}{2}[(tr(\bar{\boldsymbol{C}})^2 - tr(\bar{\boldsymbol{C}^2})], \quad \bar{I}_3 = \det \bar{\boldsymbol{C}} = 1.$$

FEM for nearly incompressible elastic materials

- Strain energy:

$$\Psi = \underbrace{\frac{\kappa}{4}(J-1)^2}_{\Psi_{vol}} + \underbrace{\frac{\mu}{2}(\bar{I}_1 - 3) + \frac{c}{2}(\bar{I}_2 - 3)}_{\Psi_{iso}}$$

  If $\kappa \gg \mu, c$ **nearly incompressible**. Pure displacement approximation may result in stability problems, oscilations, locking etc.

- The remedy – a 2-field formulation:

  $\boldsymbol{u}$ - displacements, p - the Lagrange multiplier.

- The constitutive relations:

$$\begin{cases} \boldsymbol{S} = -pJ\boldsymbol{C}^{-1} + J^{-2/3}\mathrm{Dev}[\bar{\boldsymbol{S}}], \\ p = -\Psi'_{vol}(J), \\ \bar{\boldsymbol{S}} = 2\dfrac{\partial \Psi_{iso}}{\partial \bar{\boldsymbol{C}}}, \end{cases}$$

  p can be interpreted as pressure,

  $\mathrm{Dev}[\bullet] := (\bullet) - \frac{1}{3}[(\bullet) : \boldsymbol{C}]\boldsymbol{C}^{-1}$ – deviatoric operator.

Mixed formulation

- Boundary-value problem (with $\boldsymbol{P} = \boldsymbol{F}\boldsymbol{S}$): find $\{\boldsymbol{u}, p\}$ such that

$$\begin{cases} -\mathrm{Div}\, \boldsymbol{P} &= \hat{\boldsymbol{b}} \quad \text{in} \quad \Omega, \\ \boldsymbol{u} &= \hat{\boldsymbol{u}} \quad \text{on} \quad \Gamma_D, \\ \boldsymbol{P}\boldsymbol{N} &= \hat{\boldsymbol{t}} \quad \text{on} \quad \Gamma_N. \end{cases}$$

- The weak formulation: find $\boldsymbol{u} \in \hat{\boldsymbol{u}} + V$, $p \in Q$ such that

$$\begin{cases} \int_\Omega \frac{1}{2}(\boldsymbol{F}^T \boldsymbol{\nabla} \boldsymbol{v} + \boldsymbol{\nabla}^T \boldsymbol{v}\, \boldsymbol{F}) : \boldsymbol{S}\, dV = \int_\Omega \hat{\boldsymbol{b}} \cdot \boldsymbol{v}\, dV + \int_{\Gamma_N} \hat{\boldsymbol{t}} \cdot \boldsymbol{v}\, dS, \quad \forall\, \boldsymbol{v} \in V, \\ \int_\Omega (p + \Psi'_{vol}(J))\, q\, dV = 0, \quad \forall\ q \in Q. \end{cases}$$

- Functional spaces: $V = \{\boldsymbol{v} \in H^1(\Omega) : \ \boldsymbol{v} = 0 \ \text{on} \ \Gamma_D\}$,
  $Q = L^2(\Omega)$.

- Finite element spaces, hexahedron (Simo-Pister-Taylor 1985):

$$V_h = \{\boldsymbol{v} \in H^1(\Omega) :\ \boldsymbol{v}_{|K} \in Q^p(K)\}, \qquad Q^p = \mathrm{span}\{\xi^i \eta^j \zeta^k :\ i, j, k \le p\},$$

$$Q_h = \{q \in L^2(\Omega) :\ q_{|K} \in P^{p-1}(K)\}, \quad P^{p-1} = \mathrm{span}\{\xi^i \eta^j \zeta^k :\ i+j+k \le p-1\}.$$

Mixed formulation – details

- Formulation with the first Piola-Kirchhoff stress tensor:

$$-\int_\Omega \frac{\partial P_{iJ}}{\partial X_J} v_i \, dV = \int_\Omega \hat{b}_i v_i \, dV$$

$$-\int_\Omega \left[\frac{\partial}{\partial X_J}(P_{iJ} v_i) - P_{iJ}\frac{\partial v_i}{\partial X_J}\right] dV = \int_\Omega \hat{b}_i v_i \, dV$$

$$-\int_\Gamma \underbrace{P_{iJ} N_J v_i}_{\hat{t}\cdot\boldsymbol{v}} \, dS + \int_\Omega P_{iJ}\frac{\partial v_i}{\partial X_J} \, dV = \int_\Omega \hat{b}_i v_i \, dV$$

$$\int_\Omega \boldsymbol{P} : \boldsymbol{\nabla v} \, dV = \int_\Omega \hat{\boldsymbol{b}} \cdot \boldsymbol{v} \, dV + \int_{\Gamma_N} \hat{\boldsymbol{t}} \cdot \boldsymbol{v} \, dS$$

- Formulation with the second Piola-Kirchhoff stress tensor:

$$\int_\Omega P_{iJ}\frac{\partial v_i}{\partial X_J} \, dV = \int_\Omega \underbrace{P_{mJ}\frac{\partial X_K}{\partial x_m}}_{S_{JK}} \overbrace{\phantom{\frac{\partial X_K}{\partial x_m}}}^{\delta_{mi}} \underbrace{\frac{\partial x_i}{\partial X_K}}_{F_{iK}} \underbrace{\frac{\partial v_i}{\partial X_J}}_{(\boldsymbol{\nabla v})_{iJ}} \, dV = \int_\Omega \frac{1}{2}(\boldsymbol{F}^T \boldsymbol{\nabla v} + \boldsymbol{\nabla}^T \boldsymbol{v} \boldsymbol{F}) : \boldsymbol{S} \, dV$$

$$\int_\Omega \frac{1}{2}(\boldsymbol{F}^T \boldsymbol{\nabla v} + \boldsymbol{\nabla}^T \boldsymbol{v} \, \boldsymbol{F}) : \boldsymbol{S} \, dV = \int_\Omega \hat{\boldsymbol{b}} \cdot \boldsymbol{v} \, dV + \int_{\Gamma_N} \hat{\boldsymbol{t}} \cdot \boldsymbol{v} \, dS, \ \ \forall \, \boldsymbol{v} \in V$$

The idea of error estimation for nonlinear elasticity: Rüter, Stein, CMAME 2000

- Rüter, Stein 2000: Helmholtz decomposition of 1st Piola-Kirchhoff tensor:
  $\boldsymbol{P} - \boldsymbol{P}_h = \boldsymbol{\nabla}\boldsymbol{\psi} + \boldsymbol{\nabla} \times \boldsymbol{\phi}$, and $(\boldsymbol{\nabla} \times \boldsymbol{\phi})\boldsymbol{N} = \boldsymbol{0}$ on Γ.

- Representation of the equilibrium residual:

$$R(\boldsymbol{v}) = \underbrace{\int_\Omega \hat{\boldsymbol{b}} \cdot \boldsymbol{v} + \int_{\Gamma_N} \hat{\boldsymbol{t}} \cdot \boldsymbol{v}}_{\int_\Omega P:\nabla v} - \int_\Omega \boldsymbol{P}_h : \boldsymbol{\nabla}\boldsymbol{v} \equiv \int_\Omega (\boldsymbol{P} - \boldsymbol{P}_h) : \boldsymbol{\nabla}\boldsymbol{v} = (\boldsymbol{\psi}, \boldsymbol{v})_1$$

- Find $\boldsymbol{\psi}$: $(\boldsymbol{\psi}, \boldsymbol{v})_1 = R(\boldsymbol{v}),\ \forall\, \boldsymbol{v} \in H(\Omega)$ – bvp for error function $\boldsymbol{\psi}$.

- $|\boldsymbol{\psi}|_1$ is estimated with **some error estimation technique**:
  $\boldsymbol{\psi}_K$: $(\boldsymbol{\psi}_K, \boldsymbol{v})_{1,K} = R_K(\boldsymbol{v})\ \forall \boldsymbol{v}$, $\sum_K R_K = R$.
  $|\boldsymbol{\psi}|_1 \leq (\sum_K |\boldsymbol{\psi}_K|^2_{1,K})^{1/2}$.

- The total error:

$$|\boldsymbol{u} - \boldsymbol{u}_h|_1 + \|p - p_h\|_0 \leq C \left[|\boldsymbol{\psi}|_1 + \left(\sum_K \|\Psi'_{vol} + p_h\|^2_{0,K} \right)^{1/2} \right]$$

Weak formulation for Laplace equation

We show the error estimation techniques for a scalar Laplace equation as estimates for subsequent components of the flux **decouple**.

The weak formulation of Laplace equation:

strong: **variational:**

$$
\left.\begin{array}{rcl}
-\boldsymbol{\nabla}\cdot a\boldsymbol{\nabla}u & = & f \;\; \text{in} \;\; \Omega \\
\boldsymbol{u} & = & \hat{u} \;\; \text{on} \;\; \Gamma_D \\
a\dfrac{\partial u}{\partial n} & = & \hat{g} \;\; \text{on} \;\; \Gamma_N
\end{array}\right\}
\Longleftrightarrow
\left\{\begin{array}{l}
\text{Find} \;\; \boldsymbol{u} \in V + \hat{u}: \\
B(u,v) = L(v) \;\; \forall v \in V \\
B(u,v) = \displaystyle\int_\Omega a \; \boldsymbol{\nabla}u \cdot \boldsymbol{\nabla}v \; dx \\
L(\boldsymbol{v}) = \displaystyle\int_\Omega fv \; dx + \int_{\Gamma_N} \hat{g}v \; dS \\
V = \{\boldsymbol{v} \in H^1(\Omega): \;\; v = 0 \;\; \text{on} \;\; \Gamma_D\}.
\end{array}\right.
$$

Residual error estimate: Bank, Weiser, Math. Comp. 1985, Demkowicz, Oden, Strouboulis, CMAME 1984

- We define, in an enriched space of shape functions $V_{h,p+1}(K)$, the kernel of the interpolation operator Π_{hp}:

$$M_K = \{v \in V_{h,p+1}(K): \quad \Pi_{hp} v = 0\}. \tag{0.1}$$

- Next, we formulate the local boundary-value problems in the kernel M_K: find $\phi_K(\boldsymbol{x}) \in M_K$ such that:

$$B(\phi_K, v) = r_K(v), \ \forall v \in M_K.$$

The residual on the right-hand side is defined as:

$$r_K(\boldsymbol{v}) = \int_K (f + \boldsymbol{\nabla} \cdot a \boldsymbol{\nabla} u_h) \ v \ dx + \int_{\partial K \cap \Gamma_N} (\hat{g} - a \frac{\partial u_h}{\partial n}) \cdot \boldsymbol{v} \ dS + \int_{\partial K \setminus \partial\Omega} 1/2 \overbrace{\left[\left| a \frac{\partial u_h}{\partial n} \right|\right]}^{\text{flux jump}} \cdot \boldsymbol{v} \ dS.$$

- The error estimate reads:

$$\|u_h - u\|_E^2 \leq C \sum_K \|\phi_K\|_{E,K}^2,$$

where C is a parameter growing moderately with p.

Error estimation with self-equlibration of residuals,

Ainsworth, Oden, Numer. Math. 1993

- We represent the global residual functional as the following sum of element constributions:
$$r(u_h,v) = B(u_h,v) - L(v) = \sum_K (B_K(u_h,v) - L_K(v)) =$$
$$\sum\nolimits_K \underbrace{(B_K(u_h,v) - L_K(v) - \lambda_K(v))}_{r_K(v)} = \sum_K r_K(v), \text{ with } \sum_K \lambda_K(v) = 0.$$
- In addition, we construct the functonals λ_K in such a way that:
$$r_K(1) = 0, \text{ for all elements } K - \underline{\textbf{self-equilibration}}.$$
- The last condition allows one to solve the local **Neumann** problems of the form: find $\phi_K \in V(K) = H^1(K)$ such that:
$$B_K(\phi_K, v) = r_K(v), \ \forall v \in V(K).$$
- This results in the following estimate:
$$\|u_h - u\|_E^2 \ \le \ \sum\nolimits_K \|\phi_K\|_{E,K}^2.$$

  Note that there is no auxiliary constant scaling the estimate.

Error estimation with self-equlibration of residuals

- Find $\phi_k \in H^1(K)$ such that

$$B_K(\phi_K, v) = r_K(v),\ \forall v \in H^1(K), \tag{0.2}$$

- The estimate:

$$|r(u_h, v)| =$$

$$\left|\sum_K r_K(v)\right| = \left|\sum_K B_K(\phi_K, v_{|K})\right| \leq \left|\sum_K B_K(\phi_K, \phi_K)^{1/2} B_K(v_{|K}, v_{|K})^{1/2}\right| \leq$$

$$\left(\sum_K \|\phi_K\|_{E,K}^2\right)^{1/2} \underbrace{\left(\sum_K \|v_{|K}\|_{E,K}^2\right)^{1/2}}_{\|v\|_E}$$

- The energy norm of the error $=$ the dual norm of the residual:

$$\|u_h - u\|_E = \|r\|_* = \sup_{v \in V} \frac{|r(u_h, v)|}{\|v\|_E} \leq \left(\sum_K \|\phi_K\|_{E,K}^2\right)^{1/2}.$$

Self-equilibration due to Ladeveze and Maunder

CMAME 1996

- We define the average flux between elements K and L over their common face:
$$\bar{t}_K = \frac{1}{2}((a\boldsymbol{\nabla} u_h)^K + (a\boldsymbol{\nabla} u_h)^L) \cdot \boldsymbol{n}_K.$$
- Next, we wish to find a function $\theta_K(s)$ to satisfy the following conditions for all 8 vertices of element (Ladeveze called it: **prolongation conditions**):

$$\underbrace{B_K(u_h, \psi_K^n) - L_K(\psi_K^n) - \overbrace{\int_{\partial K} \bar{t}_K \psi_K^n \, dS}}_{G_K^n} \overbrace{+ \int_{\partial K} \theta_K \psi_K^n \, dS}^{\lambda_K(\psi_K^n)} = 0, \quad n = 1, \ldots, 8.$$

Here G_K^n is a known part of the element residual, $\theta_K(s)$ is to be found.

- We postulate that $\theta_K(s)$ is a linear combination of 4 bilinear functions on each face f of K:
$$\theta_{K|f}(s) = \sum_{n=1}^{4} \theta_n^f \cdot \psi_{K|f}^n(s).$$
where θ_n^f are the coefficients to be found.

Typical situation: the rectangular mesh

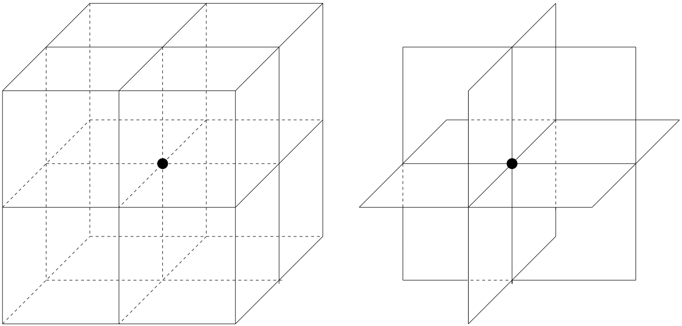


Figure: Typical situation: 8 hexes surrounding one vertex and the corresponding 12 walls adjacent to this vertex

Self-equilibration due to Ladeveze and Maunder

- The prologation conditions read:

$$\int_{\partial K} \theta_K(s)\, \psi_K^n(s)\, dS = G_K^n, \quad n = 1, \ldots, 8.$$

In this way we reduced the problem of finding $\theta_K(s)$ to obtain for each element K, 4×6 parameters θ_n^f, such that functions $\theta_{K|f}(s)$ are identical but opposite for the neigboring elements.

- Solution of the above problem can be obtained as follows:

For a fixed vertex n of element K, let the unknown contributions to the integral on the left-hand side be denoted $g_K^{n,f}$:

$$\int_{\partial K} \theta_K \psi_K^n\, dS = \sum_{f \in \mathsf{supp}(\psi_K^n)} g_K^{n,f}, \quad \text{where} \quad g_K^{n,f} = \int_f \theta_K \psi_K^n\, dS.$$

- With this notation the prologation condition takes the form:

$$\sum_{f \in \mathsf{supp}(\boldsymbol{\psi}^n)} g_K^{n,f} = G_K^n, \quad n = 1, \ldots, 8.$$

Self-equilibration due to Ladeveze and Maunder

- We assume that vertex n belongs to N neighboring elements K ($N = 8$ in the example).
- Each element has 3 faces adjacent to vertex n, and each face is common to 2 elements.
- This means we have $3/2N$ independent parameters d_i corresponding to unknown coefficients θ_n^f (we have $3/2 \cdot 8 = 12$ in the example).
- Let's denote by $i(K,f)$ appropriate indices of d_i, and by $\mathsf{sgn}(K,f) = \pm 1$ the corresponding signs connecting d_i and $g_K^{n,f}$:

$$g_K^{n,f} = d_{i(K,f)} \cdot \mathsf{sgn}(K,f), \ \forall K \in \mathsf{supp}(\boldsymbol{\psi}^n).$$

- With this notation we state the prolongation conditions as follows:

$$\sum_{f \in \mathsf{supp}(\psi^n) \cap \partial K} d_{i(K,f)} \cdot \mathsf{sgn}(K,f) = G_K^n, \ \forall K \in \mathsf{supp}(\boldsymbol{\psi}^n)$$

  or in the matrix form: $\boldsymbol{A}\boldsymbol{d} = \boldsymbol{G}$.

  The above system of N equations with $3/2N$ unknows is underdetermined.

Self-equilibration due to Ladeveze and Maunder

- The explicit form is (for an example rectangular mesh):

$$
\begin{bmatrix}
+1 & & & +1 & & & & & +1 & & & \\
-1 & +1 & & & & & & & & +1 & & \\
 & -1 & +1 & & & & & & & & +1 & \\
 & & -1 & -1 & & & & & & & & +1 \\
 & & & & +1 & & & & -1 & & & \\
 & & & & -1 & +1 & & +1 & & -1 & & \\
 & & & & & -1 & +1 & & & & -1 & \\
 & & & & & & -1 & -1 & & & & -1
\end{bmatrix}
\begin{bmatrix} d_1 \\ d_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ d_{12} \end{bmatrix}
=
\begin{bmatrix} G_1 \\ G_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ G_8 \end{bmatrix}
$$

- We can solve it by considering the minimization of function $\boldsymbol{d}^T\boldsymbol{M}\boldsymbol{d}$, with $\boldsymbol{M}$ being symmetric positive definite

$$
\left\{
\begin{array}{rcl}
\frac{1}{2}\boldsymbol{d}^T\boldsymbol{M}\boldsymbol{d} & = & \min, \\
\boldsymbol{A}\boldsymbol{d} & = & \boldsymbol{G},
\end{array}
\right.
$$

- Having found d_i for all faces attached to the corners n, we can find parameters θ_n^f:

$$
g_K^{nf} = \int_f \underbrace{\left(\sum_{m=1}^{4} \theta_m^f \psi_{K|f}^m \right)}_{\theta_{K|f}} \psi_{K|f}^n \, dS, \quad n = 1, \ldots, 4. \quad \rightarrow \quad g_K^{nf} = \sum_{i=1}^{4} \mu_{mn} \theta_m^f
$$

where: $\mu_{mn} = \int_f \psi_{K|f}^m \; \psi_{K|f}^n \; dS$, is the mass-like matrix.

Self-equilibration due to Ainsworth/Oden, Numer. Math. 1993

- In this method we consider a similar condition as in Ladeveze:

$$\underbrace{B_K(u_h,\psi^n) - L_K(\psi^n) - \int_{\partial K\setminus\partial\Omega} \bar{t}_K \psi^n \, dS}_{G_K^n} - \int_{\partial K\setminus\partial\Omega} \psi^n(s)\cdot\mu_{KL}(s) \left[\left| a\frac{\partial u_h}{\partial n}\right|\right] dS = 0.$$

  where functions $\mu_{KL}(s)$, are to be found between element K and L, with $\mu_{KL} = -\mu_{LK}$.
- Functions $\mu_{KL}(s)$ are found as follows. First we define the quantity:

$$\rho_{KL}^n := \int_{\Gamma_{KL}} \psi^n(s) \left[\left| a\frac{\partial u_h}{\partial n}\right|\right] dS, \quad \text{"jump of the flux."}$$

- Next, we solve for coefficients μ_{KL}^n:

$$\sum_L \underbrace{\mu_{KL}^n \rho_{KL}^n}_{\hat{\mu_{KL}}} = G_K^n \quad \rightarrow \quad \sum_L \underbrace{\hat{\mu}_{KL}^n = G_K^n}_{\text{Like in Ladeveze!}} \quad \rightarrow \quad \hat{\mu}_{KL}^n \quad \rightarrow \quad \mu_{KL}^n = \hat{\mu}_{KL}^n / \rho_{KL}^n$$

- Finally, the functions $\mu_{KL}(s)$ are expressed by the above coefficients μ_{KL}^n as follows:

$$\mu_{KL}(s) := \sum_{n=1}^{8} \psi^n(s)\mu_{KL}^n$$

  It can be verified that $\mu_{KL}(s)$ satisfy the prolongation conditions with 1 replacing ψ^n:

$$G_K^n - \int_{\partial K\setminus\partial\Omega} 1\cdot\mu_{KL}(s) \left[\left| a\frac{\partial u_h}{\partial n}\right|\right] dS = 0, \quad \text{i.e. equlibrium of } r_K.$$

Examples: errors on unconstrainded meshes - tube

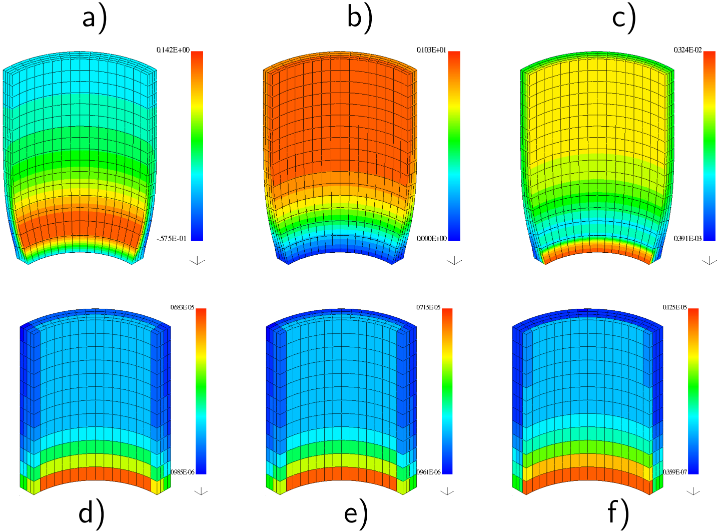


Figure: Tube pumped by internal pressure. Solution: a) uz, b) u_r, c) σ_0. Errors: d) Ainsworth/Oden, e) Ladeveze/Maunder, f) Demkowicz *et al.*

Errors on unconstrained meshes - “tyre tread”

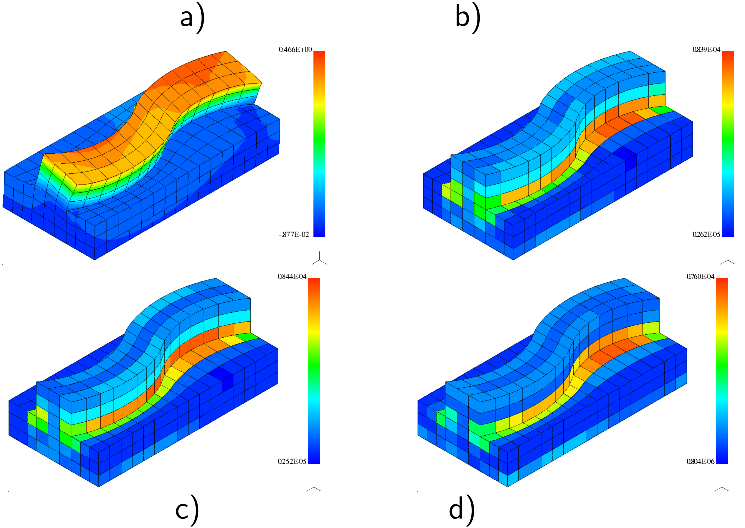
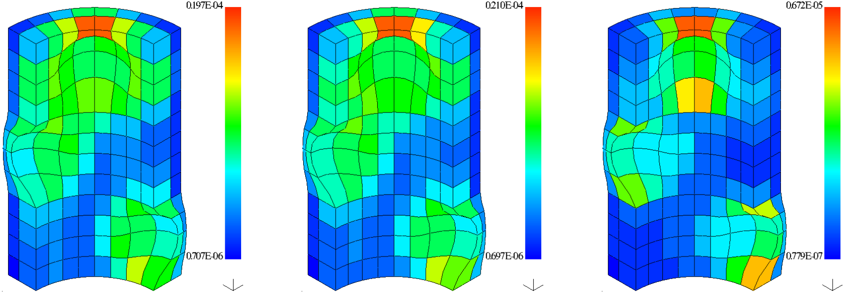


Figure: “Tyre tread.” a) solution u_y (on deformed configuration). Errors: b) Ainsworth/Oden, c) Ladeveze/Maunder, d) Demkowicz *et al.*

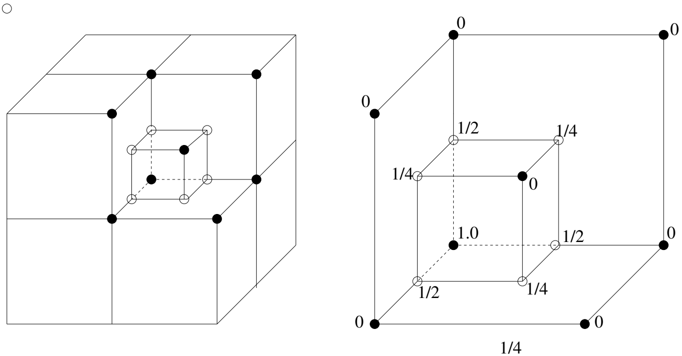
Errors on unconstrained meshes - irregular tube



a) b) c)

Figure: Irregular tube.
Errors: a) Ainsworth/Oden, b) Ladeveze/Maunder, c) Demkowicz *et al.*

Generalization for constrained meshes



$$\Phi_i = \sum_j R_{ij} \phi_j$$

Figure: Illustration of a constrained shape function Φ_i associated with a central vertex node.

Coefficients R_{ij} expressing the constrained shape function Φ_i by ordinary shape functions ϕ_j are found automatically for every element

Constrained walls in Ladeveze/Maunder method

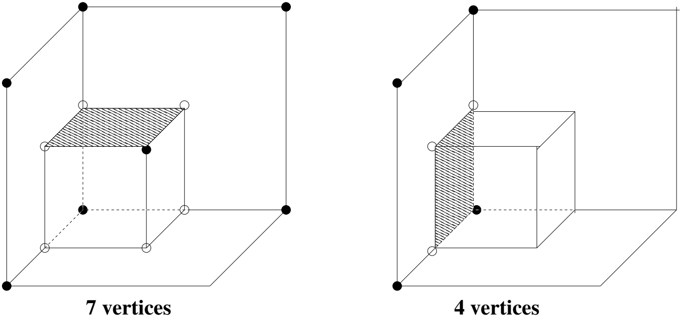


Figure: Constrained walls where value of approximation depends on:
a) values in 7 vertices - excluded in the algorithm,
b) values in 4 vertices - included in the algorithm

Ainsworth/Oden method –
just replace ordinary shape functions by constrained shape functions:

$$\phi_i \quad \to \quad \Phi_i = \sum_j R_{ij}\phi_j.$$

Algorithm

```
for vert=1,8
    find elements attached to ivert, list(1:nrel)

    for i=1,nrel
        for wall=1,6
            collect and classify faces attached to ivert
        endfor wall
    endfor i

    for i=1,nrel
        find augmented residual of element G_K^n
        for wall=1,6
            build equation for face A_{i,j}
        endfor wall
    endfor i

    solve system of eq. for d_i (or μ_KL)
endfor vert
```

Develop equilibrated residual $\boldsymbol{r}_K$

Solve Neumann problem for ϕ_K

Evaluate error e_K

Errors on constrained meshes - "tyre tread"

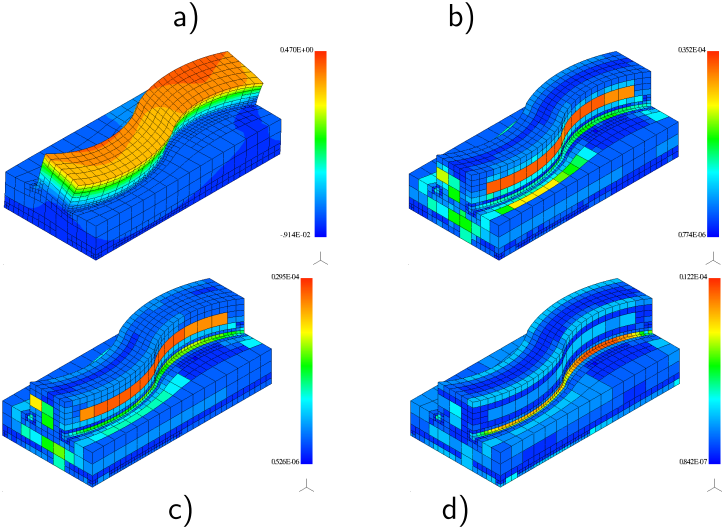
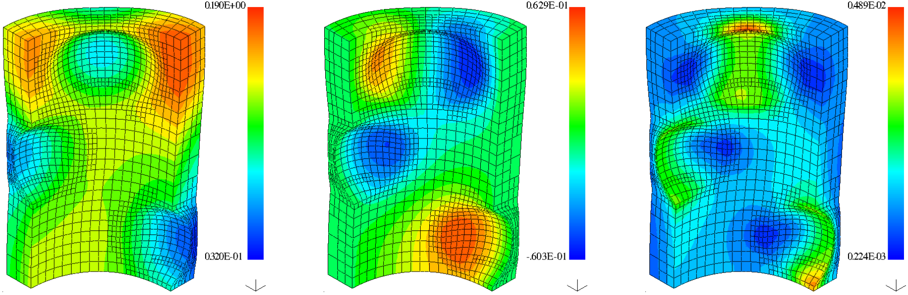


Figure: "Tyre tread." a) solution u_y.
Errors: b) Ainsworth/Oden, c) Ladeveze/Maunder, d) Demkowicz *et al.*

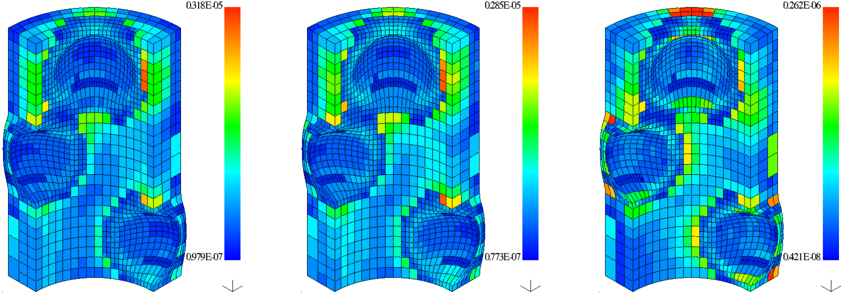
Solution of pumped irregular tube



a) b) c)

Figure: Irregular tube. Solution componets: a) u_r, b) u_θ, c) σ_0 (on deformed configuration)

Errors on constrained meshes - irregular tube



a) b) c)

Figure: Irregular tube.
Errors: a) Ainsworth/Oden, b) Ladeveze/Maunder, c) Demkowicz *et al.*

Correlation of the estimates

$$\rho_{xy} = \frac{1}{n-1}\sum_{i=1}^{n}\frac{(x_i-\bar{x})(y_i-\bar{y})}{S_x S_y}$$

$$S_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2, \quad \bar{x} = \frac{1}{n}\sum_{i=1}^{n}x_i$$

$$S_y^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i-\bar{y})^2, \quad \bar{y} = \frac{1}{n}\sum_{i=1}^{n}y_i$$

Problem	mesh	Ainsworth/Oden Ladeveze/Maunder	Ainsworth/Oden Demkowicz *et al.*	Demkowicz *et al.* Ladeveze/Maunder
tube strainght	h0	0.99	0.95	0.94
“tire tread”	h0	0.99	0.89	0.89
	h1	0.96	0.75	0.83
	h2	0.93	0.57	0.72
tube irregular	h0	0.96	0.89	0.88
	h1	0.87	0.64	0.82
	h2	0.93	0.70	0.77
	h3	0.91	0.67	0.74

Table: Correlation between the errors

Summary and conclusions

- We presented a study of applying the top-valued error estimation techniques to FE solutions of nonlinear elasticity:

  i) the implicit method of Demkowicz/Oden/Strouboulis (or Bank/Weiser),

  ii) self-equilibration method of Ainsworth/Oden (in two versions).
- The self-equilibration method has a unique feature of lacking of any auxiliary constant scaling the estimate.
- We described the algorithms working for the 1-irregular meshes in 3D.
- We observed a good correlation of the errors computed with the considered methods on a number of representative meshes, both rectagular and curvilinear.

  This correlation is especially excellent for regular meshes.
- The last fact indicates that, even if one relies on solving problems on dense uniform meshes, using large computational resources, it is worth to estimate the error to prove the reliability of the simulation.
- Of course, the error estimation techniques can serve for the adaptive methods, in case when large computational resaurces are not available.